Claremont Center for the Mathematical Sciences

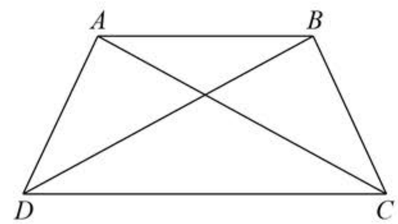
CCMS Math Challenge - H Problem Set 4 - February 1, 2016

High school students are invited to submit solutions by the end of the month in which the challenge is first published. The problems are **difficult**; an easier set is also available.

Details at: **ccms.claremont.edu/mc**

1. Let A and B be points in the plane at distance 2 from each other. Let S be the set of points P such that $(PA)^2 + (PB)^2$ is at most 10. What is the area of S?

2. Let $ABCD$ be a trapezoid, with $\overline{AB}$ parallel to $\overline{CD}$. Draw diagonals $\overline{AC}$ and $\overline{BD}$ and assume that $\angle DAC = \angle DBC$. Show that $AC = BD$.



3. Let us say that an infinite set S of positive integers is *anticlosed* if whenever x and y are two different members of S, their sum $x + y$ is not a member of S. Prove that the set of all positive integers is the union of some infinite collection of anticlosed subsets. Decide whether or not the set of all positive integers is the union of two anticlosed subsets.

4. Let us say that a set of three or more prime numbers is *amazing* if the sum of every three of them is also a prime number. For example, the set $\{5, 7, 11, 181\}$ is an amazing set of primes. Prove that no amazing set of four primes can contain 3 and that no amazing set of five primes exists.

5. Suppose $\square$ is an operation that defines a new integer $x \square y$ whenever integers x and y are given. Assume that this operation satisfies the following conditions for all nonnegative integers x and y: (a) $1 \square 0 = 1$, (b) $(2x) \square x = 2(x \square x)$ and (c) $(x+1) \square y = (x \square y) + (y^2+1) \square 0$. Compute $5 \square 20$.

6. Let x and y be nonnegative real numbers. Prove that

$$4(x^9 + y^9) \geq (x^2 + y^2)(x^3 + y^3)(x^4 + y^4)$$